Lyapunov exponent, K-S entropy and Charged Myers Perry Spacetimes

Parthapratim Pradhan[1]

Department of Physics
Vivekananda Satabarshiki Mahavidyalaya
Manikpara, Paschim Medinipur
West Bengal 721513, India

Abstract

We compute the principal Lyapunov exponent and *Kolmogorov-Sinai(K-S) entropy* for charged Myers Perry black hole space-times and investigate the instability of the equatorial circular geodesics, both massive and massless particles via these exponents. We also show that for more than four space time dimensions($N \geq 3$), there are *no* Innermost Stable Circular Orbit (ISCO) in charged Myers Perry black hole spacetime. The other aspect we have studied that among the all possible circular geodesics, which encircle the central black-hole, the time like circular geodesics has the *longest* possible orbital period i.e. $T_{timelike} > T_{null}$, than the null circular geodesics (photon sphere) as measured by asymptotic observers. Thus, the time like circular geodesics provide the *slowest way* to circle the charged Myers Perry black-hole.

PACS number(s): 04.70.Bw, 05.50.+h

1 Motivation

Geodesics especially equatorial circular geodesics in four dimensional $(3+1)$ spacetime[14, 15, 18, 19, 20] are extensively studied. But in $(N+1)$ space time dimension, such studies are greatly important both from theoretical and phenomenological point of view. Since this types of geodesics around a black hole playing a central role in general relativity for classification of the orbits. It also finds important feature of the spacetimes and gives important information on the background geometry.

It has been shown that[4] higher dimensional black hole space-time have certain number of distinguished features than the four dimension spacetime. For example, the higher dimensional black hole space-time possesses black-ring, black-string and black-Saturn solutions, whereas four dimensional space-time does not have any such solutions. Again

[1]E-mail: pppradhan77@gmail.com, pppradhan@vsm.org.in

the higher dimensional space-time has different horizon topologies, whereas in $(3+1)$ space-time the horizon topologies are 2-sphere. Therefore a phase transition occur when one going from $(3+1)$ dimension space-time to $(N+1)$ dimension space-time.

Since the Einstein equations are non-linear therefore the higher dimensional Einstein equations are also nonlinear. This properties makes the nonlinearity of higher dimensional Einstein's general theory of relativity. So, there may be a connection between nonlinear higher dimensional Einstein's general theory of relativity and non linear dynamics. Particularly, the Lyapunov exponent and Kolmogorov Sinai(K-S) entropy are two of the bridges between them. We have already computed the Lyapunov exponent and Kolmogorov Sinai entropy in terms of the ISCO equation for Reissner-Nordstrøm blackhole[18] and for Kerr-Newman black hole[20]. Here we compute the Lyapunov exponent and K-S entropy for charged Myers Perry blackhole space-time. Using it we shall prove that for more than four space-time dimensions ($N \geq 3$), there is no Innermost Stable Circular Orbit (ISCO) in charged Myers Perry blackhole space-time.

The Letter is organized as follows: in section2 we give the basic definition of Lyapunov exponent and also show that it may be expressed in terms of the radial effective potential. In section 3 we provide the relation between Lyapunov exponent and K-S entropy. In section 4 we describe reciprocal of Critical exponent can be expressed in terms of the effective radial potential. In section 5, we fully describe the equatorial circular geodesics, both time-like and null case for charged Myers Perry space-times. In section 6 the Lyapunov exponent can be used to study the instability of time-like circular geodesics. In section 7 we compute the reciprocal of the Critical exponent and we finally conclude with discussions in section 10.

2 Lyapunov exponent:

In any classical phase space the Lyapunov exponent gives a measure of the average rates of expansion and contraction of a trajectories surrounding it. They are the key indicators of chaos in any dynamical systems and also they are the asymptotic quantities defined locally in state space, and describe the exponential rate at which a perturbation to a trajectory of a system grows or decays with time at a certain location in the state space. A positive Lyapunov exponent indicates a divergence between two nearby geodesics, i.e. the paths of such a system are extremely sensitive to changes of the initial conditions. A negative Lyapunov exponent implies a convergence between two nearby geodesics. Lyapunov exponents can also distinguish among fixed points, periodic motions, quasi-periodic motions, and chaotic motions.

In classical physics, an n-dimensional autonomous smooth dynamical system is governed by the differential equation[24] of the form

$$\frac{dx}{dt} = F(x; M) \ . \tag{1}$$

where t is defined usually as time parameter. Following [9], chaos may be quantified in terms of Lyapunov exponents when the following prescriptions are maintained: a) the system is autonomous; b) the relevant part of the phase space is bounded; c) the invariant measure is normalizable; d) the domain of the time parameter is infinite. This definition signifies that the Lyapunov exponent is invariant under space diffeomorphisms of the form $u = \psi(x)$. As a result, chaos is a property of the physical system and does not depend on the coordinates used to describe the system.

In general relativity(GR), there is no concept of absolute time, therefore the time parameter forces us to consider equation (1) under spacetime diffeomorphism: $u = \psi(x)$, $d\tau = \eta(x)dt$. Thus the classical indicators of chaos like Lyapunov exponent and K-S entropy explicitly depend on the choice of the time parameter. This non-invariant characterization implies that chaos is a property of the coordinate system rather than a property of the physical system. Motivated by the work of Motter[9], we find that chaos, as characterized by positive Lyapunov exponents and positive Kolmogorov -Sinai entropy. They are coordinate invariants and transform according to

$$\lambda_i^\tau = \frac{\lambda_i^t}{< \eta >_t} . \tag{2}$$

and

$$h_{ks}^\tau = \frac{h_{ks}^t}{< \eta >_t} . \tag{3}$$

where $0 < (< \eta >_t) < \infty$ is the time average of $\eta = \frac{d\tau}{dt}$ over typical trajectory and $i = 1, ..., n$, n is the phase-space dimension. Transformation like $u = \psi(x)$, $d\tau = \eta(x)dt$ is composed of a time re-parametrization followed by a space diffeomorphism. It is well known that the Lyapunov exponents and Kolmogorov-Sinai entropy are invariant under space diffeomorphism[23]. In our previous works [18, 20], we have analysed in detail the derivation of Lyapunov exponents using proper time. Following this the Lyapunov exponent may be expressed in terms of the radial potential

$$\lambda = \pm\sqrt{\frac{(\dot{r}^2)''}{2}} . \tag{4}$$

where we may defined $\dot{r}^2$ as radial potential or effective radial potential. In general the Lyapunov exponent come in $\pm$ pairs to conserve the volume of phase space. The circular orbit is unstable when the λ is real, the circular orbit is stable when the λ is imaginary and the circular orbit is marginally stable when $\lambda = 0$.

3 Kolmogorov-Sinai entropy and Lyapunove exponent:

An important quantity which is related to the Lyapunov exponents is so called Kolmogorov-Sinai [1] entropy (h_{ks}), gives a measure of the amount of information lost or gained by a trajectory as it evolves. Following Pesin [3] it is equal to the sum of the positive Lyapunov exponents i.e

$$h_{ks} = \sum_{\lambda_i > 0} \lambda_i \ . \tag{5}$$

In 2-dimensional phase-space, there are two Lyapunov exponent, since h_{ks} is equal to the sum of positive Lyapunov exponent, here the Kolmogorov-Sinai entropy in terms of effective radial potential is given by

$$h_{ks} = \sqrt{\frac{(\dot{r}^2)''}{2}} \ . \tag{6}$$

This entropy is some sense different from the physical or statistical entropy, for example the entropy of the 2nd law of thermodynamics or blackhole entropy. Formally it is defined somewhat like entropy in statistical mechanics i.e it involves a partition of phase space.

4 Critical exponent and Radial potential:

Following Pretorius and Khurana[11], we can define Critical exponent which is the ratio of Lyapunov time scale T_λ and Orbital time scale T_Ω may be written as

$$\gamma = \frac{\Omega}{2\pi\lambda} = \frac{T_\lambda}{T_\Omega} = \frac{Lyapunov\,Timescale}{Orbital\,Timescale} \ . \tag{7}$$

where we have introduced $T_\lambda = \frac{1}{\lambda}$ and $T_\omega = \frac{2\pi}{\Omega}$, which is important for black-hole merger in the ring down radiation. In terms of the square of the proper radial velocity ($\dot{r}^2$), Critical exponent can be written as

$$\gamma = \frac{T_\lambda}{T_\Omega} = \frac{1}{2\pi}\sqrt{\frac{2\Omega^2}{(\dot{r}^2)''}} \ . \tag{8}$$

Alternatively the reciprocal of critical exponent is proportional to the effective radial potential which is given by

$$\frac{1}{\gamma} = \frac{T_\Omega}{T_\lambda} = 2\pi\sqrt{\frac{(\dot{r}^2)''}{2\Omega^2}} \ . \tag{9}$$

5 Instability of Equatorial Circular Geodesics of the Charged Myers-Perry space-time:

We will start our journey with a charged Myers-Perry black hole of $N+1$ dimension which rotates in a single plane with only one non-zero angular momentum parameter a and is a solution of the vacuum Einstein equation. Therefore the space-time metric in terms of Boyer-Lindquist coordinates is given by [10, 5, 13]

$$\begin{aligned} ds^2 &= -\left(1 - \frac{mr^{4-N}}{\Sigma} + \frac{q^2\, r^{2(3-N)}}{\Sigma}\right) dt^2 - \frac{2a\left(mr^{4-N} - q^2\, r^{2(3-N)}\right)\sin^2\theta}{\Sigma}\, dt\, d\phi \\ &+ \left(r^2 + a^2 + \frac{a^2\left(mr^{4-N} - q^2\, r^{2(3-N)}\right)\sin^2\theta}{\Sigma}\right)\sin^2\theta\, d\phi^2 + \frac{r^{N-2}\,\Sigma}{\Delta}\, dr^2 \\ &+ \Sigma\, d\theta^2 + r^2\cos^2\theta\, d\Omega^2_{N-3}\ , \end{aligned} \tag{10}$$

where,

$$\Delta = r^{N-2}(r^2 + a^2) - mr^2 + q^2 r^{4-N}, \tag{11}$$

$$\Sigma = r^2 + a^2\cos^2\theta, \tag{12}$$

$$d\Omega^2_{N-3} = d\chi_1^2 + \sin^2\chi_1[d\chi_2^2 + \sin^2\chi_2(\cdots d\chi^2_{N-3})]. \tag{13}$$

The electromagnetic potential one form for the space-time (10) is

$$A = A_\mu dx^\mu = -\frac{Q}{(N-2)r^{N-4}\Sigma}(dt - a\sin^2\theta d\phi). \tag{14}$$

The determinant (g) of the metric (10) gives

$$\sqrt{-g} = \sqrt{\gamma}\Sigma r^{N-3}\sin\theta\cos^{N-3}\theta, \tag{15}$$

where γ is the determinant of the metric (13).

The parameters $m,\ a,\ q$ are related with the physical mass (M), angular momentum (J) and charge (Q) through the relations given by

$$M = \frac{A_{N-1}(N-1)}{16\pi G}m. \tag{16}$$

$$J = \frac{A_{N-1}}{8\pi G}ma. \tag{17}$$

$$Q^2 = \frac{(N-2)(N-1)A_{N-1}}{8\pi G}q^2. \tag{18}$$

Here A_{N-1} is the area of the unit sphere in $N-1$ dimensions

$$A_{N-1} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \cos^{N-3}\theta d\theta \prod_{i=1}^{N-3} \int_0^{\pi} \sin^{(N-3)-i} \chi_i d\chi_i = \frac{2\pi^{N/2}}{\Gamma(N/2)}. \tag{19}$$

The position of the event horizon is represented by the largest root of the polynomial $\Delta_{r=r_+} = 0$. The angular velocity at the event horizon is given by

$$\Omega_H = \frac{a}{r_+^2 + a^2}. \tag{20}$$

The semiclassical Bekenstein-Hawking entropy reads[13]

$$S = \frac{A_H}{4} = \frac{\pi r_+^{N-3}(r_+^2 + a^2)\bar{A}_{N-3}}{N-2}. \tag{21}$$

where

$$\bar{A}_{N-3} = \prod_{i=1}^{N-3} \int_0^{\pi} \sin^{(N-3)-i} \chi_i d\chi_i = \frac{\pi^{\frac{N}{2}-1}}{\Gamma(\frac{N}{2}-1)}. \tag{22}$$

The surface gravity of this space-times (10) is given by

$$\kappa = \frac{(N-4)(r_+^2 + a^2) + 2r_+^2 - (N-2)q^2 r_+^{2(3-N)}}{2r_+(r_+^2 + a^2)}. \tag{23}$$

and the Hawking temperature reads

$$T_H = \frac{\kappa}{2\pi} = \frac{(N-4)(r_+^2 + a^2) + 2r_+^2 - (N-2)q^2 r_+^{2(3-N)}}{4\pi r_+(r_+^2 + a^2)}. \tag{24}$$

In appropriate limits the metric (10), the BH entropy (21), surface gravity (23) and Hawking temperature (24) reproduces the $N+1$ dimensional spherically symmetric, static Schwarzscild, Reissner-Nordstrøm [2] and axially symmetric, Myers-Perry spcetime[4].

To compute the geodesics in the equatorial plane for the Charged Myers Perry space-time we follow[21]. To determine the geodesic motions of a test particle in this plane we set $\dot{\theta} = 0$ and $\theta = constant = \frac{\pi}{2}$.

Therefore the necessary Lagrangian for this motion is given by

$$\mathcal{L} = \frac{1}{2}\left[-\left(1 - \frac{m}{r^{N-2}} + \frac{q^2}{r^{2N-4}}\right)\dot{t}^2 - \left(\frac{2ma}{r^{N-2}} - \frac{2aq^2}{r^{2N-4}}\right)\dot{t}\,\dot{\phi} + \frac{r^N}{\Delta}\dot{r}^2 + \left(r^2 + a^2 + \frac{ma^2}{r^{N-2}} - \frac{a^2q^2}{r^{2N-4}}\right)\dot{\phi}^2\right]. \tag{25}$$

The generalized momenta can be derived from it are

$$p_t = -\left(1-\frac{m}{r^{N-2}}+\frac{q^2}{r^{2N-4}}\right)\dot{t}-\left(\frac{ma}{r^{N-2}}-\frac{aq^2}{r^{2N-4}}\right)\dot{\phi} = -E = Const\ . \quad (26)$$

$$p_\phi = -\left(\frac{ma}{r^{N-2}}-\frac{aq^2}{r^{2N-4}}\right)\dot{t}+\left(r^2+a^2+\frac{ma^2}{r^{N-2}}-\frac{a^2q^2}{r^{2N-4}}\right)\dot{\phi} = L = Const\ . \quad (27)$$

$$p_r = \frac{r^N}{\Delta}\dot{r}\ . \quad (28)$$

Here $(\dot{t},\ \dot{r},\ \dot{\phi})$ denotes differentiation with respect to proper time(τ). Since the Lagrangian does not depends on 't' and 'ϕ', so p_t and p_ϕ are conserved quantities. The independence of the Lagrangian on 't' and 'ϕ' manifested, the stationarity and the axisymmetric character of the Charged Myers Perry space-time. The Hamiltonian is given by

$$\mathcal{H} = p_t\,\dot{t}+p_\phi\,\dot{\phi}+p_r\,\dot{r}-\mathcal{L}\ . \quad (29)$$

In terms of the metric the Hamiltonian is

$$\begin{aligned} 2\mathcal{H} &= -\left(1-\frac{m}{r^{N-2}}+\frac{q^2}{r^{2N-4}}\right)\dot{t}^2-\left(\frac{2ma}{r^{N-2}}-\frac{2aq^2}{r^{2N-4}}\right)\dot{t}\,\dot{\phi}+ \\ &\quad \frac{r^N}{\Delta}\dot{r}^2+\left[r^2+a^2+\frac{ma^2}{r^{N-2}}-\frac{a^2q^2}{r^{2N-4}}\right]\dot{\phi}^2\ . \end{aligned} \quad (30)$$

Since the Hamiltonian is independent of 't', therefore we can write it as

$$\begin{aligned} 2\mathcal{H} &= -\left[\left(1-\frac{m}{r^{N-2}}+\frac{q^2}{r^{2N-4}}\right)\dot{t}+\left(\frac{ma}{r^{N-2}}-\frac{aq^2}{r^{2N-4}}\right)\dot{\phi}\right]\dot{t}+\frac{r^N}{\Delta}\dot{r}^2+ \\ &\quad \left[\left(\frac{ma}{r^{N-2}}-\frac{aq^2}{r^{2N-4}}\right)\dot{t}+\left(r^2+a^2+\frac{ma^2}{r^{N-2}}-\frac{a^2q^2}{r^{2N-4}}\right)\dot{\phi}\right]\dot{\phi}\ . \end{aligned} \quad (31)$$

$$= -E\,\dot{t}+L\,\dot{\phi}+\frac{r^N}{\Delta}\dot{r}^2 = \epsilon = const\ . \quad (32)$$

Here $\epsilon=-1$ for time-like geodesics, $\epsilon=0$ for light-like geodesics and $\epsilon=+1$ for space-like geodesics. Solving equations (26) and (27) for $\dot{\phi}$ and $\dot{t}$, we find

$$\dot{\phi} = \frac{r^{N-2}}{\Delta}\left[\left(1-\frac{m}{r^{N-2}}+\frac{q^2}{r^{2N-4}}\right)L+\left(\frac{ma}{r^{N-2}}-\frac{aq^2}{r^{2N-4}}\right)E\right]\ . \quad (33)$$

$$\dot{t} = \frac{r^{N-2}}{\Delta}\left[\left(r^2+a^2+\frac{ma^2}{r^{N-2}}-\frac{a^2q^2}{r^{2N-4}}\right)E-\left(\frac{ma}{r^{N-2}}-\frac{aq^2}{r^{2N-4}}\right)L\right]\ . \quad (34)$$

Inserting these solutions in equations (32), we obtain the radial equation for Charged Myers Perry space-time which is given by

$$r^2\dot{r}^2 \;=\; r^2E^2+\left(\frac{m}{r^{N-2}}-\frac{q^2}{r^{2N-4}}\right)(aE-L)^2+\left(a^2E^2-L^2\right)+\epsilon\frac{\Delta}{r^{N-2}}\;. \qquad (35)$$

5.1 Circular Null Geodesics:

For null geodesics $\epsilon=0$, introducing new quantities $m=2M$ and $q=Q$ for simplicity the radial equation (35) becomes

$$r^2\dot{r}^2 \;=\; r^2E^2+\left(\frac{2M}{r^{N-2}}-\frac{Q^2}{r^{2N-4}}\right)(aE-L)^2+\left(a^2E^2-L^2\right)\;. \qquad (36)$$

The equations finding the radius of r_c of the unstable circular 'photon orbit' at $E=E_c$ and $L=L_c$ are

$$E_c^2r_c^2+\left(\frac{2M}{r_c^{N-2}}-\frac{Q^2}{r_c^{2N-4}}\right)(aE_c-L_c)^2+\left(a^2E_c^2-L^2\right) \;=\; 0\;. \qquad (37)$$

$$2r_cE_c^2+\left(-(N-2)\frac{2M}{r_c^{N-1}}+(2N-4)\frac{Q^2}{r_c^{2N-3}}\right)(aE_c-L_c)^2 \;=\; 0\;. \qquad (38)$$

Now introducing the impact parameter $D_c=\frac{L_c}{E_c}$, the above equations may be written as

$$r_c^2+\left(\frac{2M}{r_c^{N-2}}-\frac{Q^2}{r_c^{2N-4}}\right)(a-D_c)^2+\left(a^2-D_c^2\right) \;=\; 0\;. \qquad (39)$$

$$r_c-\left((N-2)\frac{M}{r_c^{N-1}}-(N-2)\frac{Q^2}{r_c^{2N-3}}\right)(a-D_c)^2 \;=\; 0\;. \qquad (40)$$

From equation (40) we have

$$D_c \;=\; a\mp\frac{r_c^{N-1}}{\sqrt{M(N-2)r_c^{N-2}-(N-2)Q^2}}\;. \qquad (41)$$

The equation (39) is valid if and only if $\mid D_c\mid> a$. For counter rotating orbit, we have $|D_c-a|=-(D_c-a)$, which correspond to upper sign in the above equation and co-rotating $|D_c-a|=+(D_c-a)$,which correspond to lower sign in the above equation . Inserting equation (41) in (39) we find an equation for the radius of null circular orbits

$$r_c^{2N-4}-NMr_c^{N-2}\pm 2ar_c^{N-3}\sqrt{(N-2)Mr_c^{N-2}-(N-2)Q^2}+(N-1)Q^2 \;=\; 0\;. \qquad (42)$$

When $N = 3$, we recover the well known photon sphere equation for the Kerr Newman space-times[20]. Another important relation can be derived using equations (39) and (41) for null circular orbits are

$$D_c^2 \;=\; a^2 + r_c^2 \left(\frac{NMr_c^{N-2} - (N-1)Q^2}{(N-2)Mr_c^{N-2} - (N-2)Q^2} \right) . \tag{43}$$

Now we will derive an important quantity associated with the circular null geodesics is the angular frequency which is denoted by Ω_c

$$\Omega_c = \frac{\left[\left(1 - \frac{2M}{r_c^{N-2}} + \frac{Q^2}{r_c^{2N-4}}\right) D_c + \left(\frac{2M}{r_c^{N-2}} - \frac{Q^2}{r_c^{2N-4}}\right) a\right]}{\left[\left(r_c^2 + a^2 + \frac{2Ma^2}{r_c^{N-2}} - \frac{a^2Q^2}{r_c^{2N-4}}\right) - a\left(\frac{2M}{r_c^{N-2}} - \frac{Q^2}{r_c^{2N-4}}\right) D_c\right]} = \frac{1}{D_c} . \tag{44}$$

Using equations (41) and (39) we show that the angular frequency Ω_c of the circular null geodesics is inverse of the impact parameter D_c, which generalizes the result of Kerr Newmann case[20] to the charged Myers Perry black-hole space-time. It proves that this is a *general feature* of any higher dimensional stationary space-time.

5.2 Circular Timelike Geodesics:

The time-like geodesics equation (35) can be written as by setting $\epsilon = -1$

$$r^2\dot{r}^2 \;=\; r^2E^2 + \left(\frac{2M}{r^{N-2}} - \frac{Q^2}{r^{2N-4}}\right)(aE - L)^2 + \left(a^2E^2 - L^2\right) - \frac{\Delta}{r^{N-2}} . \tag{45}$$

where E is the energy per unit mass of the particle describes the trajectory.

Now we shall find the radial equation of the timelike circular geodesics in terms of reciprocal radius $u = 1/r$ as the independent variable, can be expressed as

$$\begin{aligned}\mathcal{V}(u) \;&=\; u^{-4}\dot{u}^2 = E^2 + 2Mu^N (aE - L)^2 - u^{2N-2}Q^2 (aE - L)^2 \\ &\quad + \left(a^2E^2 - L^2\right) u^2 - 1 - a^2u^2 + 2Mu^{N-2} - Q^2u^{2N-4} .\end{aligned} \tag{46}$$

The conditions for the occurrence of circular orbits at $r = r_0$ or reciprocal radius $u = u_0$ are

$$\mathcal{V}(u) \;=\; 0 . \tag{47}$$

and

$$\frac{d\mathcal{V}(u)}{du} \;=\; 0 . \tag{48}$$

Now setting $x = L_0 - aE_0$, where L_0 and E_0 are the values of energy and angular momentum for circular orbits at the radius $r_0 = \frac{1}{u_0}$. Therefore using (46, 48) we get the following equations

$$
\begin{aligned}
- x^2Q^2u^{2N-2} + 2Mx^2u_0^N - (x^2+2axE)u^2 - a^2u_0^2 - Q^2u_0^{2N-4} \\
+2Mu_0^{N-2} - 1 + E_0^2 &= 0 \; . \qquad (49)
\end{aligned}
$$

and

$$
\begin{aligned}
- (N-1)x^2Q^2u_0^{2N-3} + NMx^2u_0^{N-1} - (x^2+2axE_0)u_0 - a^2u_0 \\
-(N-2)Q^2u_0^{2N-5} + (N-2)Mu_0^{N-3} &= 0 \; . \qquad (50)
\end{aligned}
$$

Using (49, 50) we find an equation for E_0^2 as

$$
\begin{aligned}
E_0^2 &= 1 + (N-4)Mu_0^{N-2} + (N-2)Mx^2u_0^N \\
&\quad -(N-2)x^2Q^2u_0^{2N-2} - (N-3)Q^2u_0^{2N-4} \; . \qquad (51)
\end{aligned}
$$

with the aid of equation (51), equation (50) gives us

$$
\begin{aligned}
2axE_0u_0 &= x^2[NMu_0^{N-1} - (N-1)Q^2u_0^{2N-3} - u_0] - a^2u_0 - (N-2)Q^2u_0^{2N-5} \\
&\quad +(N-2)Mu_0^{N-3} \; . \qquad (52)
\end{aligned}
$$

Eliminating E_0 between these equations, we have obtained the following quadratic equation for x^2 i.e

$$
\mathcal{A}x^4 + \mathcal{B}x^2 + \mathcal{C} = 0 \; . \qquad (53)
$$

where

$$
\begin{aligned}
\mathcal{A} &= u_0^2\left[NMu_0^{N-1} - (N-1)Q^2u_0^{2N-3} - u_0\right]^2 - 4a^2u_0^2\left[(N-2)Mu_0^N - (N-2)Q^2u_0^{2N-2}\right] \\
\mathcal{B} &= -2\left[a^2u_0 + (N-2)Q^2U_0^{2N-5} - (N-2)Mu_0^{N-3}\right]\left[NMu_0^{N-1} - (N-1)Q^2u_0^{2N-3} - u_0\right] \\
&\quad -4a^2u_0^2\left[1 + (N-4)Mu_0^{N-2} - (N-3)Q^2u_0^{2N-4}\right] \\
\mathcal{C} &= \left[a^2u_0 + (N-2)Q^2u_0^{2N-5} - (N-2)Mu_0^{N-3}\right]^2
\end{aligned}
$$

The solution of this equations (53) are

$$
x^2 = \frac{-\mathcal{B} \pm \mathcal{D}}{2\mathcal{A}} \; . \qquad (54)
$$

where the discriminant of this equation is

$$\mathcal{D} \;=\; 4a\,\Delta_{u_0}\sqrt{(N-2)Mu_0^N-(N-2)Q^2u_0^{2N-2}}\;. \tag{55}$$

and

$$\Delta_{u_0} \;=\; 1+a^2u_0^2-2Mu_0^{N-2}+Q^2u_0^{2N-4}\;. \tag{56}$$

The solutions becomes simpler form by writing

$$\begin{aligned} &\left[1-NMu_0^{N-2}+(N-1)Q^2u_0^{2N-4}\right]^2 \\ &-4a^2\left[(N-2)Mu_0^N-(N-2)Q^2u_0^{2N-2}\right] \;=\; Z_+\,Z_-\;. \end{aligned} \tag{57}$$

where

$$\begin{aligned} Z_\pm \;=\; &\left[1-NMu_0^{N-2}+(N-1)Q^2u_0^{2N-4}\right]\pm \\ &2a\sqrt{\left[(N-2)Mu_0^N-(N-2)Q^2u_0^{2N-2}\right]}\;. \end{aligned} \tag{58}$$

Thus we get the solution as

$$x^2u_0^2 \;=\; \frac{-\mathcal{B}\pm\mathcal{D}}{Z_+Z_-}\;. \tag{59}$$

Thus we find

$$x^2u_0^2 \;=\; \frac{\Delta_{u_0}-Z_\mp}{Z_\mp}\;. \tag{60}$$

Again we can write

$$\Delta_{u_0}-Z_\mp \;=\; \left[au_0\pm\sqrt{(N-2)Mu_0^{N-2}-(N-2)Q^2u_0^{2N-4}}\right]^2\;. \tag{61}$$

Therefore the solutions for x thus may be written as

$$x \;=\; -\frac{\left[a\sqrt{u_0}\pm\sqrt{(N-2)Mu_0^{N-3}-(N-2)Q^2u_0^{2N-5}}\right]}{\sqrt{u_0Z_\pm}}\;. \tag{62}$$

Here the upper sign in the foregoing equations applies to counter-rotating orbits, while the lower sign applies to co-rotating orbits. Replacing the solution (62) for x in equation (51), we obtain the energy

$$E_0 = \frac{1}{\sqrt{Z_\mp}}\left[1 - 2Mu_0^{N-2} \mp a\sqrt{(N-2)Mu_0^N - (N-2)Q^2u_0^{2N-2}} + Q^2u_0^{2N-4}\right] . \quad (63)$$

and the value of angular momentum associated with the circular orbit is given by

$$L_0 = \mp\frac{1}{\sqrt{u_0 Z_\mp}}\left[(1+a^2u_0^2)\sqrt{(N-2)Mu_0^{N-3} - (N-2)Q^2u_0^{2N-5}} \pm 2aM\sqrt{u_0^{2N-3}} \mp aQ^2\sqrt{u_0^{4N-7}}\right] . \quad (64)$$

As we previously defined E_0 and L_0 followed by equations (63) and (64) are the energy and the angular momentum per unit mass of a particle describing a circular orbit of radius u.

To compute the stability of timelike circular orbit we must calculate the 2nd order derivative of effective potential with respect to u for the values of E_0 and L_0 specific to circular orbits.

Now the 2nd order derivative of effective potential becomes

$$\frac{d^2\mathcal{V}}{du^2} = (N-2)u^{N-4}\left[\left(NM - 2(N-1)Q^2u^{N-2}\right) - 2(N-3)Q^2u^{N-2} + (N-4)M\right] . \quad (65)$$

Using (60) we find

$$\frac{d^2\mathcal{V}}{du^2}|_{u=u_0} = \frac{2(N-2)u_0^{N-4}}{Z_\mp}\left[\left(NM - 2(N-1)Q^2u_0^{N-2}\right)\Delta_{u_0} - \left(4M - 4Q^2u_0^{N-2}\right)Z_\mp\right] \quad (66)$$

The 2nd order derivative of effective potential shows that it explicitly depends on spacetime dimensionality N. So, to determine the stability of equatorial circular geodesics we must check the sign of 2nd order derivative of the function $\mathcal{V}(u)$ which will be helpful to distinguish between different values of N. Since E_0, L_0 and $x = L_0 - aE_0$ must be real, the function Δ_u and $Z_\mp$ are such that

$$\Delta_{u_0} \geq Z_\pm \geq 0 . \quad (67)$$

Case I: For $N \geq 4$ i.e five dimensional case, the above equation leads to

$$\left[NM - 2(N-1)Q^2u_0^{N-2}\right]\Delta_{u_0} \geq \left[4M - 4Q^2u_0^{N-2}\right]Z_\mp . \quad (68)$$

which immediately suggests that

$$\frac{d^2\mathcal{V}}{du^2} \geq 0 . \quad (69)$$

Thus we conclude that there are no ISCO and stable timelike circular orbit around the rotating five-dimensional charged Myers-Perry blackhole space-time, at least in the "equatorial" planes. Which generalizes the previous work by Frolov and Stojkovic[8] on five dimensional rotating black hole.

Case II: Now in four dimension $N = 3$, the above equation (66) reduces to

$$\frac{d^2\mathcal{V}}{du^2}|_{u=u_0} = \frac{2}{u_0 Z_\mp}\left[\left(3M - 4Q^2 u_0\right)\Delta_{u_0} - \left(4M - 4Q^2 u_0\right) Z_\mp\right] . \tag{70}$$

For $\Delta_{u_0} \geq Z_\pm \geq 0$, the equation (70) leads to

$$\left(3M - 4Q^2 u_0\right)\Delta_{u_0} < \left(4M - 4Q^2 u_0\right) Z_\mp . \tag{71}$$

which implies

$$\frac{d^2\mathcal{V}}{du^2} < 0 . \tag{72}$$

Thus we proved that there exists ISCO and stable timelike circular orbit around the rotating four dimensional Kerr-Newman space-time[20].

Case III. For $N \geq 3$ i.e arbitrary dimension, the above equation (66) leads to

$$\left[NM - 2(N-1)Q^2 u_0^{N-2}\right]\Delta_{u_0} \geq \left[4M - 4Q^2 u_0^{N-2}\right] Z_\mp . \tag{73}$$

which immediately suggests that

$$\frac{d^2\mathcal{V}(u)}{du^2} \geq 0 . \tag{74}$$

Thus we conclude that in space-time dimensions greater than four i.e $N \geq 3$, there are no ISCOs and stable timelike circular orbits around the rotating higher dimensional charged Myers-Perry blackhole space-time, at least in the "equatorial" planes. Which generalizes the previous work by Cardoso[12] on higher dimensional Myers-Perry blackhole space-time.

This may suggested that the absence of bounded stable circular orbit in the black hole exterior is a generic property of higher dimensional charged black holes. Which generalizes the previous work by Tangerlini[2] on non-rotating higher dimensional black hole and by Frolov and Stojkovic[8] on five dimensional rotating black hole.

5.2.1 Angular velocity of Timelike Circular Orbit

Now we compute the orbital angular velocity for timelike circular geodesics at $r = r_0$ is given by

$$\Omega_0 = \frac{\dot{\phi}}{\dot{t}} = \frac{\left[\left(1 - \frac{2M}{r_0^{N-2}} + \frac{Q^2}{r_0^{2N-4}}\right) L_0 + \left(\frac{2M}{r_0^{N-2}} - \frac{Q^2}{r_0^{2N-4}}\right) a E_0\right]}{\left[\left(r_0^2 + a^2 + \frac{2Ma^2}{r_0^{N-2}} - \frac{a^2 Q^2}{r_0^{2N-4}}\right) E_0 - a\left(\frac{2M}{r_0} - \frac{Q^2}{r_0^2}\right) L_0\right]} . \tag{75}$$

Again this can be rewritten as

$$\Omega_0 = \frac{\left[L_0 - 2Mu_0^{N-2}x + Q^2u_0^{2N-4}x\right] u_0^2}{(1+a^2u_0^2)E_0 - 2axu_0^2(2Mu_0^{N-2} - Q^2u_0^{2N-4})} \ . \tag{76}$$

Now the previously mentioned expression can be simplified as

$$L_0 - 2Mxu_0^{N-2} + Q^2xu_0^{2N-4} = \mp\frac{\sqrt{(N-2)Mu_0^{N-3} - (N-2)Q^2u_0^{2N-5}}}{\sqrt{u_0}Z_\mp}\Delta_{u_0} \ . \tag{77}$$

$$(1+a^2u_0^2)E_0 - 2aMxu_0^N + axQ^2u_0^{2N-2} = \frac{\Delta_{u_0}}{Z_\mp}\left[1 \mp a\sqrt{(N-2)Mu_0^N - (N-2)Q^2u_0^{2N-2}}\right] \tag{78}$$

Substituting (77) and (78) into (76) we get the angular velocity for timelike circular geodesics is given by

$$\Omega_0 = \mp\frac{\sqrt{(N-2)Mu_0^N - (N-2)Q^2u_0^{2N-2}}}{1 \mp a\sqrt{(N-2)Mu_0^N - (N-2)Q^2u_0^{2N-2}}} \ . \tag{79}$$

5.2.2 Ratio of Angular velocity of time like circular orbit to Null Circular Orbit

Since we have already proved that for timelike circular geodesics the angular velocity is given by the equation (79)

Again we obtained for circular null geodesics $\Omega_c = \frac{1}{D_c}$, so we can deduce similar expression for it is given by

$$\Omega_c = \mp\frac{\sqrt{(N-2)Mu_c^N - (N-2)Q^2u_c^{2N-2}}}{1 \mp a\sqrt{(N-2)Mu_c^N - (N-2)Q^2u_c^{2N-2}}} \ . \tag{80}$$

Resultantly we obtain the ratio of angular frequency for time-like circular geodesics to the angular frequency for null circular geodesics is

$$\frac{\Omega_0}{\Omega_c} = \left(\frac{\sqrt{Mr_0^{N-2} - Q^2}}{\sqrt{Mr_c^{N-2} - Q^2}}\right)\left(\frac{r_c^{N-1} \mp a\sqrt{(N-2)Mr_c^{N-2} - (N-2)Q^2}}{r_0^{N-1} \mp a\sqrt{(N-2)Mr_0^{N-2} - (N-2)Q^2}}\right) \ . \tag{81}$$

which is proportional to the radial coordinates r_0. For $r_0 = r_c$, $\Omega_0 = \Omega_c$,i.e , when the radius of time-like circular geodesics is equal to the radius of null circular geodesics, the angular frequency corresponds to that geodesic are equal, which demands that the intriguing physical phenomena could occur in the curved four dimensional space-time, for example, possibility of exciting Quasi Normal Modes(QNM) by orbiting particles,

possibly leading to instabilities of the curved space-time[12]. It would be very interesting to investigate such phenomenon occur in higher dimensional space-time.

For $r_0 > r_c$, we proved that for Schwarzschild black-hole, Reissner Nordstrøm black-hole and Kerr Black-hole the null circular geodesics have the largest angular frequency as measured by asymptotic observers than the time-like circular geodesics. We therefore conclude that null circular geodesics provide the fastest way to circle black holes[17]. This generalizes the case of axisymmetric symmetry Kerr Newmann space-time[15] to the more general case of stationary, axisymmetry charged Myers Perry blackhole space-times.

Now the ratio of time period for time-like circular geodesics to the time period for null circular geodesics is given by

$$\frac{T_0}{T_c} = \left(\frac{\sqrt{Mr_c^{N-2}-Q^2}}{\sqrt{Mr_0^{N-2}-Q^2}}\right)\left(\frac{r_0^{N-1}\mp a\sqrt{(N-2)Mr_0-(N-2)Q^2}}{r_c^{N-1}\mp a\sqrt{(N-2)Mr_c-(N-2)Q^2}}\right) . \tag{82}$$

This ratio is valid for $r_0 \neq r_c$. For $r_0 = r_c$, $T_0 = T_c$, i.e. time period of both geodesics are equal, which possibly leading to the excitations of Quasi Normal Modes. For $r_0 > r_c$, $T_0 > T_c$, which implies that the orbital period for time-like circular geodesics is greater than the orbital period for null circular geodesics. For $r_0 = r_{particle}$ and $r_c = r_{photon}$, therefore the ratio of time period for the orbit of massive particles ($r_0 = r_{timelike}$) to the time period for photon-sphere ($r_c = r_{photon}$) for charged Myers Perry black-hole is given by

$$\frac{T_{particle}}{T_{photon}} = \left(\frac{\sqrt{Mr_c^{N-2}-Q^2}}{\sqrt{Mr_0^{N-2}-Q^2}}\right)\left(\frac{r_0^{N-1}\mp a\sqrt{(N-2)Mr_0-(N-2)Q^2}}{r_c^{N-1}\mp a\sqrt{(N-2)Mr_c-(N-2)Q^2}}\right) . \tag{83}$$

This implies that $T_{particle} > T_{photon}$, therefore we conclude that timelike circular geodesics provide the *slowest way* to circle the charged Myers Perry black-hole space-time.

Here we may note that we recover from (42) the condition for the occurrence of the well known unstable circular null geodesics by taking the limit $E_0 \to \infty$, when

$$\begin{aligned} Z_\pm &= \left[1 - NMu_0^{N-2} + (N-1)Q^2u_0^{2N-4}\right] \pm \\ &\quad 2a\sqrt{\left[(N-2)Mu_0^N - (N-2)Q^2u_0^{2N-2}\right]} = 0 . \end{aligned} \tag{84}$$

or alternatively for $r_0 = r_c$

$$r_c^{2N-4} - NMr_c^{N-2} \pm 2ar_c^{N-3}\sqrt{(N-2)Mr_c^{N-2}-(N-2)Q^2} + (N-1)Q^2 = 0 . \tag{85}$$

Here $(-)$ sign indicates for direct orbit and $(+)$ sign indicates for retrograde orbit. The real positive root of the equation is the closest circular photon orbit of the blackhole.

6 Lyapunov exponent and Timelike Circular Geodesics:

Now we evaluated the Lyapunov exponent and K-S entropy for timelike circular geodesics as follows, using equations (4) we get

$$\lambda_{CMP} = h_{K-S} = \sqrt{\frac{(N-2)}{r_0^{N-2} Z_{\mp 0}} [r_0^{N-4}(NMr_0^{N-2} - 2(N-1)Q^2)\Delta_{r_0} - (4Mr_0^{N-2} - Q^2)Z_{\mp 0}]} \; . \tag{86}$$

where

$$\begin{aligned} \Delta_{r_0} &= r_0^N + a^2 r_0^{N-2} - 2Mr_0^2 + Q^2 r_0^{4-N} \\ Z_{\mp 0} &= r_0^{2N-4} - NMr_0^{N-2} \mp 2ar_0^{N-3}\sqrt{(N-2)Mr_0^{N-2} - (N-2)Q^2} + (N-1)Q^2 \end{aligned} \tag{87}$$

Since $\Delta_{r_0} \geq Z_{\mp 0} \geq 0$ and for $r_0^{N-4}(NMr_0^{N-2} - 2(N-1)Q^2)\Delta_{r_0} \geq (4Mr_0^{N-2} - 4Q^2)Z_{\mp 0}$, λ is real, so we conclude that there are no ISCOs around the charged Myers-Perry blackhole space-time.

7 Critical exponent and Timelike Circular geodesics:

Now We determine the Critical exponent of charged Myers Perry blackhole space-time for equatorial timelike circular geodesics. From that we shall prove the instability of equatorial timelike circular geodesics via Critical exponent. Therefore the reciprocal of Critical exponent is given by

$$\frac{1}{\gamma} = 2\pi \frac{\sqrt{\left[r_0^{N-4}(NMr_0^{N-2} - 2(N-1)Q^2)\Delta_{r0} - (4Mr_0^{N-2} - 4Q^2)Z_{\mp 0}\right] (r_0^{N-1} \mp a\sqrt{(N-2)Mr_0 - (N-2)Q}}}{\sqrt{r_0^{N-2}(Mr_0^{N-2} - Q^2)Z_{\mp 0}}}$$

Since $Z_{\mp 0} \geq 0$, $\Delta_{r_0} \geq 0$ and $(NMr_0^{N-2} - 2(N-1)Q^2)\Delta_{r_0} \geq (4Mr_0^{N-2} - 4Q^2)Z_{\mp 0}$, so $\frac{1}{\gamma}$ is real, which also implies that equatorial time like circular geodesics is unstable.

8 Lyapunov exponent and Null Circular Geodesics:

For null circular geodesics the Lyapunov exponent and K-S entropy of charged Myers Perry blackhole are given by

$$\lambda_{CMP}^{null} = h_{ks}^{CMP} = \sqrt{\frac{(N-2)(L_c - aE_c)^2[NMr_c^{N-2} - 2(N-1)Q^2]}{r_c^{2N}}} \; . \tag{89}$$

Since $NMr_c^{N-2} > 2(N-1)Q^2$ therefore λ is real so the null circular geodesics are unstable. In the appropriate limits, we can obtain the Lyapunov exponent and K-S entropy for Myers Perry space-times, Tangherlini RN space-times and Tangherlini Schwarzschild space-times.

9 Critical Exponent and Null Circular geodesics:

Therefore the reciprocal of Critical exponent is given by

$$\frac{1}{\gamma} = 2\pi\sqrt{\frac{(L_c - aE_c)^2[NMr_c^{N-2} - 2(N-1)Q^2](r_c^{N-1} \mp a\sqrt{(N-2)Mr_c - (N-2)Q^2})^2}{r_c^{2N}(Mr_c^{N-2} - Q^2)}} . \quad (90)$$

10 Discussion

The study demonstrates that Lyapunov exponent and Kolmogorov-Sinai entropy may be used to give a full description of time-like circular geodesics and null circular geodesics in charged Myers Perry black hole space-time. We showed that the Lyapunov exponent can be used to determine the instability of equatorial circular geodesics, both time-like and null case for charged Myers Perry black hole space-time. By computing Lyapunov exponent, We proved that for more than four space time dimensions($N \geq 3$), there is no Innermost Stable Circular Orbit (ISCO) in charged Myers Perry black hole spacetime. The other point we have studied that for circular geodesics around the central black-hole, time-like circular geodesics is characterized by the smallest angular frequency as measured by the asymptotic observers-no other circular geodesics can have a smallest angular frequency. Thus such types of space-times always have $\Omega_{particle} < \Omega_{photon}$ for all time-like circular geodesics. Alternatively it was shown that the orbital period for time-like circular geodesics is characterized by the longest orbital period than the null circular geodesics. Hence, we conclude that time-like geodesics provide the *slowest way* to circle the black hole.

References

[1] A. N. Kolmogorov, *Dokl. Akkad. Nauk. SSSR* **98**, 527 (1954); Ya. G. Sinai, *Dokl. Akkad. Nauk. SSSR* **124**, 768 (1959).

[2] F. R. Tangherlini, Nuovo Cimento **27**, 636 (1963).

[3] Ya. B. Pesin, *Dokl. Akad. Nauk. SSSR.* **226**, 774 (1976).

[4] R. C. Myers, M. J. Perry, *Ann. Phys.* **172**, 304 (1986).

[5] X. Dianyan, *Class. Quant. Grav.* **5**, 871 (1988).

[6] N. J. Cornish, *Phys. Rev.* **D 64**, 084011 (2001) [arXiv: gr-qc/0106062].

[7] N. J. Cornish and J. J. Levin, *Class. Quant. Grav.* **20**, 1649(2003) [arXiv: gr-qc/0304056].

[8] V. P. Frolov and D. Stojkovic, *Phys. Rev.* **D 67**, 084004 (2003).

[9] A. E. Motter, *Phys. Rev. Lett.* **91**, 231101 (2003).

[10] A. N. Aliev, Phys. Rev. **D 74**, 024011 (2006).

[11] F. Pretorius and D. Khurana, *Class. Quant. Grav.* **24**, S83 (2007) [arXiv: gr-qc/0702084].

[12] V. Cardoso, A. S. Miranda, E. Berti, H. Witek, V. T. Zanchin *Phys. Rev.* **D 79** (2009) 064016.

[13] R. Banerjee, B. R. Majhi, S. K. Modak, S. Samanta, *Phys. Rev.* **D 82**:124002, 2010

[14] P. Pradhan, P. Majumdar, *Physics Letters* **A 375** (2011) 474-479; arXiv: 1001.3582 [gr-qc].

[15] P. Pradhan, P. Majumdar, Extremal Limits and Kerr Space-time; arXiv:1108.2333v1 [gr-qc].

[16] M. R. Setare and D. Momeni, Int J Theor Phys **50**, 106-113 (2011).

[17] S. Hod, *Phys. Rev.* **D 84** (2011) 104024.

[18] P. Pradhan, Lyapunov Exponent and Reissner Nordstrøm Black Hole; arXiv:1205.5656[gr-qc].

[19] P. Pradhan, *Journal of Physics: Conference Series* **405** (2012) 012027.

[20] P. Pradhan, ISCO, Lyapunov exponent and Kerr-Newman Black Hole; arXiv: 1212.5758 [gr-qc].

[21] S. Chandrashekar, *The Mathematical Theory of Black Holes*, Clarendon Press, Oxford (1983).

[22] A. M. Lyapunov, *The General Problem of the Stability of Motion*, Taylor and Francis, London (1992).

[23] E. Ott, *Chaos in Dynamical Systems*, Cambridge University Press, Cambridge (1994).

[24] A. H. Nayfeh and B. Balachandran, *Applied Nonlinear Dynamics* , Wiley-Vch Verlag GmbH & Co., (2004).